

On the Deterministic Code Capacity Region of an Arbitrarily Varying Multiple-Access Channel Under List Decoding

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Abstract—We study the capacity region C_L of an arbitrarily varying multiple-access channel (AVMAC) for deterministic codes with decoding into a list of a fixed size L and for the average error probability criterion. Motivated by known results in the study of fixed size list decoding for a point-to-point arbitrarily varying channel, we define for every AVMAC whose capacity region for random codes is nonempty, a nonnegative integer U called its symmetrizability. It is shown that for every $L \leq U$, C_L has an empty interior, and for every $L \geq (U+1)^2$, C_L equals the capacity region of the AVMAC for random codes with a known single-letter characterization.

I. INTRODUCTION

We study the deterministic code capacity region of an arbitrarily varying multiple-access channel (AVMAC) under fixed size- L list decoding. For every received sequence, a list decoder outputs a list of message pairs of size at most L . The error occurs when the transmitted message pair is not in the output list. We restrict ourselves to a discrete memoryless AVMAC with finite inputs, output and state alphabets and the average error probability criterion.

For point-to-point transmission over an arbitrarily varying channel (AVC), it is known [1] that the (list-of-1 size) deterministic code capacity equals either 0 or the random code capacity. The latter capacity is defined for a “random code” in which the encoder and the decoder are assumed to have shared access to a random experiment whose result can be used in selecting a deterministic code from a pool of such codes. Later, it was shown in [6], [5] that a necessary and sufficient condition for the deterministic code capacity to be zero is when the AVC is “symmetrizable.” When list decoding of a fixed size L is considered, it also holds that the list-of- L size capacity for deterministic codes equals either 0 or the random code capacity; a necessary and sufficient condition for the list-of- L size capacity for deterministic codes to be zero was given in [3], [8] in terms of a quantity termed the “symmetrizability” of the AVC defined in [8]. This latter concept can be regarded as a generalization of the symmetrizable condition of the AVC considered in [6], [5]. Precisely, an AVC is symmetrizable if its symmetrizability is at least 1.

Next, turning to transmission over an AVMAC with the usual decoding ($L = 1$), Jahn [9] has shown that the capacity region C_1 for deterministic codes either has an empty interior

or equals the random code capacity region defined and characterized therein. Gubner [7] introduced the symmetrizable condition for an AVMAC and showed that it implies that the interior of C_1 is empty. Later, Ahlswede and Cai [2] proved that this condition is also necessary for C_1 to have an empty interior.

In the present paper, we introduce a concept of symmetrizability of an AVMAC and study its relationship with its list-of- L size capacity region for deterministic codes

II. PRELIMINARIES

We start with the definition of a discrete memoryless AVMAC and certain quantities showing its specific behavior.

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and \mathcal{S} be finite sets representing the alphabets of the two inputs, output and state, respectively. The AVMAC is determined by a family of conditional distributions $W(z|x, y, s)$ on \mathcal{Z} ($z \in \mathcal{Z}$), defined by two input symbols $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and a state $s \in \mathcal{S}$. It is assumed that the AVMAC is memoryless, i.e., that the transition probability function $W^n(\mathbf{z}|\mathbf{x}, \mathbf{y}, \mathbf{s})$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{X}^n$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathcal{Y}^n$, $(z_1, \dots, z_n) \in \mathcal{Z}^n$, $\mathbf{s} = (s_1, \dots, s_n) \in \mathcal{S}^n$ satisfies $W^n(\mathbf{z}|\mathbf{x}, \mathbf{y}, \mathbf{s}) = \prod_{i=1}^n W(z_i|x_i, y_i, s_i)$. We denote such a channel as $G = (W, \mathcal{X}, \mathcal{Y}, \mathcal{S}, \mathcal{Z})$. A deterministic code $(\mathcal{U}^{(n)}, \mathcal{V}^{(n)})$ of length n and cardinalities M_1, M_2 is a collection of $\mathcal{U}^{(n)} = \{\mathbf{x}_1, \dots, \mathbf{x}_{M_1}\} \subseteq \mathcal{X}^n$ and $\mathcal{V}^{(n)} = \{\mathbf{y}_1, \dots, \mathbf{y}_{M_2}\} \subseteq \mathcal{Y}^n$. We call $R_1 = \frac{1}{n} \log_2 M_1$ (resp. $R_2 = \frac{1}{n} \log_2 M_2$) the *rate* of the code for transmitter 1 (resp. transmitter 2) and (R_1, R_2) the *rate-tuple* of the code.

In the present paper, we consider list decoding of a fixed size L ; the usual decoding corresponds to a special case of $L = 1$. Given M_1 and M_2 , a list-of- L size decoder is defined with a collection $\{A_{ij}\}_{i=1, j=1}^{M_1, M_2}$ with $A_{ij} \subseteq \mathcal{Z}^n$, $i = 1, \dots, M_1$, $j = 1, \dots, M_2$, representing the set of all received sequences each having the message pair (i, j) in its decoding list. Further, it is required that $\bigcap_{(i,j) \in K} A_{ij} = \emptyset$ for all $K \subseteq [M_1] \times [M_2]$ with $|K| \geq L + 1$, where $[M_1] \triangleq \{1, \dots, M_1\}$, $[M_2] \triangleq \{1, \dots, M_2\}$. Consequently, a received sequence \mathbf{z} is decoded into a list of $L' \leq L$ message pairs $\{(i_1, j_1), \dots, (i_{L'}, j_{L'})\} \subseteq [M_1] \times [M_2]$. The decoding rule is $\phi_L(\mathbf{z}) = \{(i, j) \in [M_1] \times [M_2] : \mathbf{z} \in A_{ij}\}$. The code together with the decoder $C_L = (\mathcal{U}, \mathcal{V}, \phi_L)$ is called a deterministic code decoded into a list of size L . The error probability of decoding into a list of size L

when the message pair (i, j) is transmitted over the AVMAC in the state $\mathbf{s} \in \mathcal{S}^n$ is defined as

$$e_L(i, j, \mathbf{s}) = e_L(i, j, \mathbf{s}, \mathcal{C}_L) \triangleq \sum_{\mathbf{z} \in \mathcal{Z}^n: (i, j) \notin \phi_L(\mathbf{z})} W^n(\mathbf{z} | \mathbf{x}_i, \mathbf{y}_j, \mathbf{s}) \quad (1)$$

and the corresponding average error probability is defined as

$$\bar{e}_L(\mathbf{s}) = \bar{e}_L(\mathbf{s}, \mathcal{C}_L) \triangleq \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \frac{1}{M_1 M_2} e_L(i, j, \mathbf{s}). \quad (2)$$

For $R_1 > 0, R_2 > 0$, we are interested in the quantity

$$\bar{p}_L(R_1, R_2) = \limsup_{n \rightarrow \infty} \min_{\substack{\mathcal{C}_L, \log_2 M_1 \geq R_1 n \\ \log_2 M_2 \geq R_2 n}} \max_{\mathbf{s} \in \mathcal{S}^n} \bar{e}_L(\mathbf{s}, \mathcal{C}_L).$$

Define the *list-of- L size capacity region* $C_L = C_L(G)$ of G for deterministic codes under the average error probability criterion to be the closure of the region $\{(R_1 \geq 0, R_2 \geq 0) : \bar{p}_L(R_1, R_2) = 0\}$; let $\text{int}(C_L)$ denote the interior of C_L .

The random code capacity region C^R , defined in [9], will play a central role in this paper. C^R was characterized therein as the closure of the convex hull of the following region

$$\bigcup_{\substack{P_X(x) \\ P_Y(y)}} \left\{ (R_1, R_2) : \begin{array}{l} 0 \leq R_1 \leq \inf_{P_S(s)} I(X \wedge Z | Y) \\ 0 \leq R_2 \leq \inf_{P_S(s)} I(Y \wedge Z | X) \\ R_1 + R_2 \leq \inf_{P_S(s)} I(X, Y \wedge Z) \end{array} \right\},$$

with the union being over all distributions P_X on \mathcal{X} and P_Y on \mathcal{Y} and with the joint distribution of $(X, Y, Z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ being $P_{XYZ}(x, y, z) = P_X(x)P_Y(y) \sum_{s \in \mathcal{S}} P_S(s)W(z | x, y, s)$.

III. MAIN RESULTS

The following proposition extends the statement of Jahn [9] from $L = 1$ to the case $L \geq 1$.

Proposition 1: *Either C_L equals C^R or $\text{int}(C_L) = \emptyset$.*

The proof of Proposition 1 follows the derivation in [9]. When $\text{int}(C_L) \neq \emptyset$, a deterministic prefix code can be used with decoding into a list of size L at a rate-tuple with each individual rate being nonzero to show that $C_L \supseteq C^R$. The fact that $C_L \subseteq C^R$ also follows, upon noting that $C_1 \subseteq C^R$, in a similar manner to the AVC case [3], [8] and an exercise in [4, p. 230].

Definition 1: For a positive integer u , an AVMAC G is *u -symmetrizable* if either of the following holds.

a) There exists a conditional distribution

$U(s | x_2, y_2, \dots, x_{u+1}, y_{u+1}), s \in \mathcal{S}$,
 $(x_2, y_2, \dots, (x_{u+1}, y_{u+1}) \in \mathcal{X} \times \mathcal{Y}$ such that for any $x_1, \dots, x_{u+1} \in \mathcal{X}$, $y_1, \dots, y_{u+1} \in \mathcal{Y}$, $z \in \mathcal{Z}$ and any permutation π on $[u+1] \triangleq \{1, \dots, u+1\}$,

$$\sum_{s \in \mathcal{S}} W(z | x_1, y_1, s) U(s | x_2, y_2, \dots, x_{u+1}, y_{u+1}) = \sum_{s \in \mathcal{S}} W(z | x_{\pi(1)}, y_{\pi(1)}, s) U(s | x_{\pi(2)}, y_{\pi(2)}, \dots, x_{\pi(u+1)}, y_{\pi(u+1)}). \quad (3)$$

b) For some a, b satisfying $(a+1)(b+1) \geq u+1$, there exists a conditional distribution

$U(s | x_2, \dots, x_{a+1}, y_2, \dots, y_{b+1}), s \in \mathcal{S}$, $x_2, \dots, x_{a+1} \in \mathcal{X}$, $y_2, \dots, y_{b+1} \in \mathcal{Y}$ such that for any $x_1, \dots, x_{a+1} \in \mathcal{X}$, $y_1, \dots, y_{b+1} \in \mathcal{Y}$, $s \in \mathcal{S}$, $z \in \mathcal{Z}$, and any permutations π on $[a+1]$ and σ on $[b+1]$,

$$\sum_{s \in \mathcal{S}} W(z | x_1, y_1, s) U(s | x_2, \dots, x_{a+1}, y_2, \dots, y_{b+1}) = \sum_{s \in \mathcal{S}} W(z | x_{\pi(1)}, y_{\sigma(1)}, s) U(s | x_{\pi(2)}, \dots, x_{\pi(a+1)}, y_{\sigma(2)}, \dots, y_{\sigma(b+1)}). \quad (4)$$

To simplify terminology, we take all AVMACs to be 0-symmetrizable. It is clear that if G is u -symmetrizable, then G is also u' -symmetrizable for all $0 \leq u' \leq u$. The symmetrizability of G denoted by $U = U(G)$ is defined as the largest integer u for which G is u -symmetrizable.

Theorem 2: *For an AVMAC with symmetrizability U , $\text{int}(C_L) = \emptyset$ for every $L \leq U$.*

Theorem 3: *For an AVMAC with symmetrizability U and for every $L \geq (U+1)^2$, C_L equals C^R .*

IV. OUTLINE OF PROOFS

For a positive integer M , let $[M]$ denote $\{1, \dots, M\}$. For a set $K \subseteq [M] \times [M]$, let $I_K \triangleq \{i \in [M] : \exists j \in [M] \text{ such that } (i, j) \in K\}$ and $J_K \triangleq \{j \in [M] : \exists i \in [M] \text{ such that } (i, j) \in K\}$.

Outline of the Proof of Theorem 2: For a fixed $L \leq U$ and any $\delta > 0$, we consider any deterministic code (decoded into a list of size L) $\mathcal{C}_L = (\mathcal{U}^{(n)} = \{\mathbf{x}_i\}_{i=1}^M, \mathcal{V}^{(n)} = \{\mathbf{y}_i\}_{i=1}^M, \phi_L(\mathbf{z}))$ with $R = \frac{1}{n} \log_2 M \geq \delta$. By Definition 1, either (3) holds with $u = U$ or (4) holds with some a, b such that $(a+1)(b+1) = U+1$.

First, suppose that (3) holds with $u = U$. For any $K = \{(i_1, j_1), (i_2, j_2), \dots, (i_U, j_U)\} \subset [M] \times [M]$, with $(i_1, j_1) < (i_2, j_2) < \dots < (i_U, j_U)$ (for a fixed ordering of $[M] \times [M]$), satisfying $|K| = |I_K| = |J_K| = U$, let S_K denote a random state sequence with distribution $U^n(s | \mathbf{x}_{i_1}, \mathbf{y}_{j_1}, \dots, \mathbf{x}_{i_U}, \mathbf{y}_{j_U})$. Also, for any $(i, j) \in [M] \times [M]$, let

$$E[W^n(\mathbf{z} | \mathbf{x}_i, \mathbf{y}_j, S_K)] \triangleq \sum_{s \in \mathcal{S}^n} W^n(\mathbf{z} | \mathbf{x}_i, \mathbf{y}_j, s) U^n(s | \mathbf{x}_{i_1}, \mathbf{y}_{j_1}, \dots, \mathbf{x}_{i_U}, \mathbf{y}_{j_U}).$$

Then, for any $K' = \{(i_1, j_1), \dots, (i_{U+1}, j_{U+1})\} \subset [M] \times [M]$, with $(i_1, j_1) < (i_2, j_2) < \dots < (i_{U+1}, j_{U+1})$, satisfying $|K'| = |I_{K'}| = |J_{K'}| = U+1$, we have

$$\sum_{k=1}^{U+1} E[e_L(i_k, j_k, S_{K' \setminus \{(i_k, j_k)\}})] = \sum_{k=1}^{U+1} \left(1 - \sum_{\substack{\mathbf{z}: (i_k, j_k) \\ \in \phi_L(\mathbf{z})}} E[W^n(\mathbf{z} | \mathbf{x}_{i_k}, \mathbf{y}_{j_k}, S_{K' \setminus \{(i_k, j_k)\}})] \right)$$

$$\begin{aligned}
&= (U+1) - \sum_{\mathbf{z}} \sum_{\substack{k \in [U+1]: \\ (i_k, j_k) \in \phi_L(\mathbf{z})}} E[W^n(\mathbf{z} | \mathbf{x}_{i_1}, \mathbf{y}_{j_1}, S_{K' \setminus \{(i_1, j_1)\}})] \text{ by (3)} \\
&\geq (U+1) - L, \quad \text{by } |\phi_L(\mathbf{z})| \leq L. \quad (5)
\end{aligned}$$

Next, let $\mathcal{P}_U = \{K \subset [M] \times [M] : |K| = |I_K| = |J_K| = U\}$. Then, $|\mathcal{P}_U| = \binom{M}{U} U!$ and

$$\begin{aligned}
&\frac{1}{|\mathcal{P}_U|} \sum_{K \in \mathcal{P}_U} E[\bar{e}_L(S_K)] \\
&\geq \frac{1}{|\mathcal{P}_U| M^2} \sum_{K \in \mathcal{P}_U} \sum_{(i,j) \in I_K^c \times J_K^c} E[e_L(i,j, S_K)] \\
&= \frac{1}{|\mathcal{P}_U| M^2} \sum_{K' \in \mathcal{P}_{U+1}} \sum_{(i,j) \in K'} E[e_L(i,j, S_{K' \setminus \{(i,j)\}})] \\
&\geq \frac{|\mathcal{P}_{U+1}|(U+1)}{|\mathcal{P}_U| M^2} (1 - \frac{L}{U+1}), \quad \text{by (5)} \\
&= (\frac{M-U}{M})^2 (1 - \frac{L}{U+1}).
\end{aligned}$$

Then, for any $M = \lfloor 2^{\delta n} \rfloor$,

$$\liminf_{n \rightarrow \infty} \frac{1}{|\mathcal{P}_U|} \sum_{K \in \mathcal{P}_U} E[\bar{e}_L(S_K)] > 0$$

if $L \leq U$. Since the left side is an average of $\bar{e}_L(\mathbf{s})$ with respect to a distribution of S_K with K being uniform on \mathcal{P}_U and $\delta > 0$ is arbitrary small, it follows that $\text{int}(C_L) = \emptyset$.

The case in which (4) holds with some a, b satisfying $(a+1)(b+1) = U+1$ can be handled in a similar manner. In particular, by considering $\mathcal{P}_{(a,b)} = \{(I, J) : I \subset [M], J \subset [M], |I| = a, |J| = b\}$ with $|\mathcal{P}_{(a,b)}| = \binom{M}{a} \binom{M}{b}$, we can find a positive lower bound to

$$\frac{1}{|\mathcal{P}_{(a,b)}|} \sum_{(I,J) \in \mathcal{P}_{(a,b)}} E[\bar{e}_L(S_{I,J})]$$

for any $M = \lfloor 2^{\delta n} \rfloor$ where δ is arbitrarily small. Here, for $I = \{i_1, \dots, i_a\} \subset [M]$, with $i_1 < i_2 < \dots < i_a$, and $J = \{j_1, \dots, j_b\} \subset [M]$, with $j_1 < j_2 < \dots < j_b$, $S_{(I,J)}$ denotes a random state sequence with distribution $U^n(\mathbf{s} | \mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_a}, \mathbf{y}_{j_1}, \dots, \mathbf{y}_{j_b})$. The proof is omitted for this case. ■

Outline of Proof of Theorem 3: We start with some standard notations. For positive numbers a, b and some sequences, $(\mathbf{x}_1, \dots, \mathbf{x}_a, \mathbf{y}_1, \dots, \mathbf{y}_b, \mathbf{s}, \mathbf{z}) \in (\mathcal{X}^a)^n \times (\mathcal{Y}^b)^n \times \mathcal{S}^n \times \mathcal{Z}^n$, $P_{(\mathbf{x}_1, \dots, \mathbf{x}_a, \mathbf{y}_1, \dots, \mathbf{y}_b, \mathbf{s}, \mathbf{z})}$ denotes the joint type of the sequences: the empirical distribution on $\mathcal{X}^a \times \mathcal{Y}^b \times \mathcal{S} \times \mathcal{Z}$ of the sequences. For a finite set \mathcal{A} and any two distributions $P_1(a), P_2(a)$, $a \in \mathcal{A}$, let $D(P_1 || P_2)$ (resp. $d(P_1, P_2)$) denote the divergence (resp. variational distance) of P_1, P_2 .

Recall from Proposition 1 that it suffices to show that $\text{int}(C_L) \neq \emptyset$. To this end, we consider a “constant composition” code $\mathcal{U} = \mathcal{U}_{\mathbf{x}}^{(n)} = \{\mathbf{x}_1, \dots, \mathbf{x}_M\}$ and $\mathcal{V} = \mathcal{V}_{\mathbf{y}}^{(n)} = \{\mathbf{y}_1, \dots, \mathbf{y}_M\}$ with each \mathbf{x}_i (resp. \mathbf{y}_j) being of the same type $P_X(x) = P_{(\mathbf{x})}$ (resp. $P_Y(y) = P_{(\mathbf{y})}$) that coincides with the

type of a fixed sequence $\mathbf{x} \in \mathcal{X}^n$ (resp. $\mathbf{y} \in \mathcal{Y}^n$). We first describe a list decoding algorithm for such codes and show in Lemma 1 that it is a list-of- L size decoder. Then, a “good” code is specified in Lemma 4 and is used, together with the decoder, to show that $\text{int}(C_L) \neq \emptyset$.

The list decoding algorithm consists of two steps and is parameterized by a (small) parameter $\eta > 0$ to be chosen shortly. This algorithm follows the ideas of [5], [3], [8], [2].

1. Collect a list of message pairs $\Gamma \subseteq [M] \times [M]$ such that for every $(i, j) \in \Gamma$, there exists a state sequence $\mathbf{s} \in \mathcal{S}^n$ such that for $P_{XYSZ} = P_{(\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}, \mathbf{z})}$, it holds that

$$D(P_{XYSZ} || P_X \times P_Y \times P_S \times W) \leq \eta. \quad (6)$$

2. Put a message pair (i, j) in $\phi_L^{\mathcal{U}, \mathcal{V}}(\mathbf{z})$ if $(i, j) \in \Gamma$ and if for some $\mathbf{s} \in \mathcal{S}^n$ satisfying (6), it holds that for every subset $K \subseteq \Gamma$ such that $(i, j) \in K$ and $|K| = L+1$,

$$I(XYZ \wedge X^{a-1}, Y^{b-1} | S) \leq \eta, \quad (7)$$

where $a = |I_K|$ and $b = |J_K|$ and $P_{XYSZ} = P_{(\mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_{I_K \setminus \{i\}}, \mathbf{y}_{J_K \setminus \{j\}}, \mathbf{s}, \mathbf{z})}$.

Let $\phi_L^{\mathcal{U}, \mathcal{V}}(\mathbf{z}) = \{(1, 1)\}$ if no (i, j) satisfies (6) and (7).

Lemma 1: *There exists a function $f : \{0, 1, \dots\} \rightarrow \{0, 1, \dots\}$ and a sufficiently small η (cf. (6) and (7)) such that for an AVMAC with symmetrizability U , any $\alpha > 0$, any $\mathbf{x} \in \mathcal{X}^n$ (resp. $\mathbf{y} \in \mathcal{Y}^n$) satisfying $\min_{x \in \mathcal{X}} P_{(\mathbf{x})}(x) > \alpha$ (resp. $\min_{y \in \mathcal{Y}} P_{(\mathbf{y})}(y) > \alpha$), every $L \geq f(U)$ and any deterministic code $(\mathcal{U}_{\mathbf{x}}, \mathcal{V}_{\mathbf{y}})$, the decoding algorithm $\phi_L^{\mathcal{U}, \mathcal{V}}(\mathbf{z})$ as above satisfies $|\phi_L^{\mathcal{U}, \mathcal{V}}(\mathbf{z})| \leq L$ for every $\mathbf{z} \in \mathcal{Z}^n$.*

Proof: For positive integers A, M , we shall call a set $K \subseteq [M] \times [M]$ an A -diagonal if $|K| = |I_K| = |J_K| = A$ and call K an A -rectangle if $|K| = |I_K| |J_K| \geq A$. Further, K is said to contain an A -diagonal (resp. A -rectangle) if there exists $K' \subseteq K$ such that K' is an A -diagonal (resp. A -rectangle). For any positive integers A, R, M such that $A \leq R$, let $B(A, R, M) \triangleq |\{K \subseteq [M] \times [M] : |K| = R, K \text{ contains no } A\text{-diagonal and no } A\text{-rectangle}\}|$.

Claim: for any $A \geq 1$, any $R \geq (A-1)^2 A$ and any $M > 0$, $B(A, R, M) = 0$. To see this, consider an arbitrary $M > 0$ and a set $K \subseteq [M] \times [M]$ with $|K| = R \geq (A-1)^2 + 1$. If $|I_K| \geq A$ and $|J_K| \geq A$, then clearly K contains an A -diagonal. Consider the rest of the K s with, say, $|I_K| \leq A-1$. As $(A-1)^2 + 1 \leq |K| = \sum_{i \in I_K} |K \cap \{i\} \times J_K|$ and $|I_K| \leq A-1$, there exists an $i \in I_K$ for which $|K \cap \{i\} \times J_K| \geq A$, i.e., K contains an A -rectangle. This proves the claim.

Next, let

$$g(A) \triangleq \min_{R \geq A} R \leq (A-1)^2 + 1, \text{ by the Claim.} \quad (8)$$

The significance of g in (8) can be understood as follows. For $A \geq 1$ and any $M > 0$, it holds that any set $K \subseteq [M] \times [M]$ with $|K| \geq g(A)$ must contain either an A -diagonal or an A -rectangle. We now let, for every $u \geq 0$,

$$f(u) \triangleq g(u+2) - 1 \leq (u+1)^2, \text{ by (8).} \quad (9)$$

We shall prove Lemma 1 with this f by contradiction. Suppose that there exists an output sequence $\mathbf{z} \in \mathcal{Z}^n$ such that $|\phi_L^{\mathcal{U},\mathcal{V}}(\mathbf{z})| \geq L + 1$. Pick some $K \subseteq \phi_L^{\mathcal{U},\mathcal{V}}(\mathbf{z})$ with

$$|K| = L + 1 \geq f(U) + 1 = g(U + 2), \text{ by (9).} \quad (10)$$

Then, for any $(i, j) \in K$, by (6) and (7), we have that for some $\mathbf{s}_{ij} \in \mathcal{S}^n$ with $P_{X_{I_K} Y_{J_K} S_{ij} Z} = P_{(\mathbf{x}_{I_K}, \mathbf{y}_{J_K}, \mathbf{s}_{ij}, \mathbf{z})}$, it holds that

$$\begin{aligned} 2\eta &\geq D(P_{X_i Y_j S_{ij} Z} \| P_{X_i} \times P_{Y_j} \times P_{S_{ij}} \times W) \\ &\quad + I(X_i Y_j Z \wedge X_{I_K \setminus \{i\}} Y_{J_K \setminus \{j\}} | S_{ij}) \\ &= D \left(\frac{P_{X_{I_K} Y_{J_K} S_{ij} Z}}{P_{X_i} \times P_{Y_j} \times P_{S_{ij} X_{I_K \setminus \{i\}} Y_{J_K \setminus \{j\}}} \times W} \right). \end{aligned} \quad (11)$$

Next, from (10) and (8), K contains either a $(U + 2)$ -diagonal or a $(U + 2)$ -rectangle.

First, consider the case in which K contains a $(U + 2)$ -diagonal. Specifically, there exists a subset $K' \subseteq K$ such that $|K'| = |I_{K'}| = |J_{K'}| = U + 2$. By separately permuting the pair of indices of $[M] \times [M]$, we can assume without any loss of generality that $K' = \{(1, 1), \dots, (U + 2, U + 2)\}$. Applying the logsum inequality to (11) to every $(i, i) \in K'$, $i \in [U + 2]$, we get

$$2\eta \geq D \left(\frac{P_{X^{U+2} Y^{U+2} Z}}{P_{X_i} \times P_{Y_i} \times (\sum_{s \in \mathcal{S}} P_{S_{ii} X_i^{U+2} Y_i^{U+2}} \times W)} \right), \quad (12)$$

where $X_i^{U+2} \triangleq X_{[U+2] \setminus \{i\}}$ and $Y_i^{U+2} \triangleq Y_{[U+2] \setminus \{i\}}$. Applying Pinsker's inequality [4, p. 58] to (12), we get that, for each $i \in [U + 2]$,

$$c\sqrt{2\eta} \geq d \left(\frac{P_{X^{U+2} Y^{U+2} Z}}{P_{X_i} \times P_{Y_i} \times (\sum_{s \in \mathcal{S}} P_{S_{ii} X_i^{U+2} Y_i^{U+2}} \times W)} \right), \quad (13)$$

where c is an absolute constant. By the triangle inequality, we obtain that

$$2c\sqrt{2\eta} \geq \max_{1 \leq i < j \leq U+2} d \left(\frac{P_{X_i} \times P_{Y_i} \times (\sum_{s \in \mathcal{S}} P_{S_{ii} X_i^{U+2} Y_i^{U+2}} \times W)}{P_{X_j} \times P_{Y_j} \times (\sum_{s \in \mathcal{S}} P_{S_{jj} X_j^{U+2} Y_j^{U+2}} \times W)} \right). \quad (14)$$

Note that $P_{X_i} = P_{(\mathbf{x})}$ (resp. $P_{Y_i} = P_{(\mathbf{y})}$), $i = 1, \dots, U + 2$, with $\min_{x \in \mathcal{X}} P_{(\mathbf{x})}(x) > \alpha$ (resp. $\min_{y \in \mathcal{Y}} P_{(\mathbf{y})}(y) > \alpha$). The sought contradiction is obtained by invoking the following Lemma 2 upon setting η sufficiently small. The proof of Lemma 2 is similar to that of Lemma A4 of [8] and is omitted.

Lemma 2: *For an AVMAC G with symmetrizability U and any $\alpha > 0$, there exists $\nu(\alpha) > 0$ such that for any pair of distributions $P(x)$, $x \in \mathcal{X}$, and $Q(y)$, $y \in \mathcal{Y}$, satisfying $\min_{x \in \mathcal{X}} P(x) > \alpha$, $\min_{y \in \mathcal{Y}} Q(y) > \alpha$ and any collection of $U + 2$ distributions $U_i(x_i^{U+1}, y_i^{U+1}, s)$, $(x_i^{U+1}, y_i^{U+1}, s) \in \mathcal{X}^{U+1} \times \mathcal{Y}^{U+1} \times \mathcal{S}$, $i = 1, \dots, U + 2$, it holds that*

$$\max_{1 \leq i < j \leq U+2} d \left(\frac{P(x_i)Q(y_i)}{(\sum_{s \in \mathcal{S}} W(z|x_i, y_i, s)U_i(x_i^{U+2}, y_i^{U+2}, s))} \right) \geq \nu.$$

Lastly, we consider the case in which K contains a $(U + 2)$ -rectangle. Precisely, there exists $I \times J \subseteq K$ with $|I| = a + 1$, $|J| = b + 1$ and $(a + 1)(b + 1) \geq U + 2$. By separately permuting the pair of indices of $[M] \times [M]$, we can assume without any loss of generality that $I = [a + 1]$, $J = [b + 1]$. Similar to the argument leading to (14), we get

$$2c\sqrt{2\eta} \geq \max_{(i,j), (i',j') \in [a+1] \times [b+1], (i,j) \neq (i',j')} d \left(\frac{P_{X_i} \times P_{Y_j} \times (\sum_{s \in \mathcal{S}} P_{S_{ij} X_i^{a+1} Y_j^{b+1}} \times W)}{P_{X_{i'}} \times P_{Y_{j'}} \times (\sum_{s \in \mathcal{S}} P_{S_{i'j'} X_{i'}^{a+1} Y_{j'}^{b+1}} \times W)} \right). \quad (15)$$

The sought contradiction is obtained by invoking the following Lemma 3, whose proof is also omitted, upon setting η sufficiently small. This completes the proof of Lemma 1.

Lemma 3: *For an AVMAC G with symmetrizability U and any $\alpha > 0$, there exists $\nu(\alpha) > 0$ such that for any pair of distributions $P(x)$, $x \in \mathcal{X}$, and $Q(y)$, $y \in \mathcal{Y}$, satisfying $\min_{x \in \mathcal{X}} P(x) > \alpha$, $\min_{y \in \mathcal{Y}} Q(y) > \alpha$ and any collection of $(a + 1)(b + 1) \geq U + 2$ distributions $U_{ij}(x_i^a, y_j^b, s)$, $(x_i^a, y_j^b, s) \in \mathcal{X}^a \times \mathcal{Y}^b \times \mathcal{S}$, $i = 1, \dots, a + 1$, $j = 1, \dots, b + 1$, it holds that*

$$\max_{(i,j), (i',j') \in [a+1] \times [b+1], (i,j) \neq (i',j')} d \left(\frac{P(x_i)Q(y_j)}{(\sum_{s \in \mathcal{S}} W(z|x_i, y_j, s)U_{ij}(x_i^{a+1}, y_j^{b+1}, s))} \right) \geq \nu.$$

We now specify in the following Lemma 4 a “good” deterministic code, with nonzero rates; the proof of the lemma is similar to that of Lemma 2 in [2] and is omitted.

For a deterministic code $(\mathcal{U}_x^{(n)}, \mathcal{V}_y^{(n)})$ with $|\mathcal{U}| = |\mathcal{V}| = M$ and $R = \frac{1}{n} \log_2 M$, any $\epsilon > 0$ and any $\mathbf{s} \in \mathcal{S}^n$, we define

$$\mathcal{A}_\epsilon(\mathbf{s}) \triangleq \left\{ (i, j) \in [M] \times [M] : \begin{aligned} &D(P_{XY S} \| P_X \times P_Y \times P_S) < \epsilon \\ &\text{where } P_{XY S} = P_{(\mathbf{x}_i, \mathbf{y}_j, \mathbf{s})} \end{aligned} \right\}, \quad (16)$$

$$\mathcal{B}_\epsilon(\mathbf{s}) \triangleq \left\{ i \in [M] : \begin{aligned} &\text{for any } I \subseteq [M] \setminus \{i\}, |I| = L \\ &\text{and any } J \subseteq [M], |J| = L + 1, \\ &I(X \wedge X_I, Y_J, S) < (2L + 1)R + \epsilon, \\ &\text{where } P_{XX_I Y_J S} = P_{(\mathbf{x}_i, \mathbf{x}_I, \mathbf{y}_J, \mathbf{s})} \end{aligned} \right\}, \quad (17)$$

$$\mathcal{C}_\epsilon(\mathbf{s}) \triangleq \left\{ j \in [M] : \begin{aligned} &\text{for any } J \subseteq [M] \setminus \{j\}, |J| = L \\ &\text{and any } I \subseteq [M], |I| = L + 1, \\ &I(Y \wedge X_I, Y_J, S) < (2L + 1)R + \epsilon, \\ &\text{where } P_{X_I Y Y_J S} = P_{(\mathbf{x}_I, \mathbf{y}_j, \mathbf{y}_J, \mathbf{s})} \end{aligned} \right\}, \quad (18)$$

Lemma 4: *For any $0 < \epsilon < \delta$, any sequences $\mathbf{x} \in \mathcal{X}^n$ and $\mathbf{y} \in \mathcal{Y}^n$ with $H(P_{(\mathbf{x})}) > \delta$ and $H(P_{(\mathbf{y})}) > \delta$, and all sufficiently large n , there exists a deterministic code $(\mathcal{U}_x, \mathcal{V}_y)$ as above with $R \geq \delta$ such that for every $\mathbf{s} \in \mathcal{S}^n$,*

$$|\mathcal{A}_\epsilon(\mathbf{s})| \leq 2^{-\frac{\epsilon}{4}} M^2 \text{ and} \quad (19)$$

$$|\mathcal{B}_\epsilon(\mathbf{s})|, |\mathcal{C}_\epsilon(\mathbf{s})| \leq 2^{-\frac{\epsilon}{4}} M. \quad (20)$$

For a fixed $\alpha > 0$ and all n sufficiently large, choose $\mathbf{x} \in \mathcal{X}^n$ and $\mathbf{y} \in \mathcal{Y}^n$ so that $\min_{x \in \mathcal{X}} P_{(\mathbf{x})}(x) > \alpha$ and $\min_{y \in \mathcal{Y}} P_{(\mathbf{y})}(y) > \alpha$. We then choose η sufficiently small according to Lemma 1. Next, for the \mathbf{x} and \mathbf{y} , and for some ϵ and δ sufficiently small so that $H(P_{(\mathbf{x})}) > \delta$ and $H(P_{(\mathbf{y})}) > \delta$ and

$$0 < \epsilon < \delta \leq R < \frac{\eta}{2(6L+4)}, \quad (21)$$

we get from Lemma 4 a deterministic code $(\mathcal{U}_{\mathbf{x}}, \mathcal{V}_{\mathbf{y}})$ with some R satisfying (21), (19) and (20). Combining this code with the decoding algorithm from Lemma 1, we obtain a deterministic code decoded into a list of size L . Lastly, we show that for every $\mathbf{s} \in \mathcal{S}^n$, $\bar{e}_L(\mathbf{s})$ approaches zero exponentially fast.

First, we note that it suffices to prove that for all $(i, j) \in \mathcal{A}_{\epsilon}(\mathbf{s}) \cap [\mathcal{B}_{\epsilon}(\mathbf{s}) \times \mathcal{C}_{\epsilon}(\mathbf{s})]$, $e_L(i, j, \mathbf{s})$ approaches zero exponentially fast, because by (19) and (20),

$$\bar{e}_L(\mathbf{s}) \leq \frac{1}{M^2} \sum_{(i,j) \in \mathcal{A}_{\epsilon}(\mathbf{s}) \cap [\mathcal{B}_{\epsilon}(\mathbf{s}) \times \mathcal{C}_{\epsilon}(\mathbf{s})]} e_L(i, j, \mathbf{s}) + 3 \times 2^{-\frac{\epsilon}{4}n}.$$

For a fixed \mathbf{s} and $(i, j) \in \mathcal{A}_{\epsilon}(\mathbf{s}) \cap [\mathcal{B}_{\epsilon}(\mathbf{s}) \times \mathcal{C}_{\epsilon}(\mathbf{s})]$, $e_L(i, j, \mathbf{s})$ is the probability of the event

$$\bigcap_{\mathbf{s}' \in \mathcal{S}^n} \left\{ E_0(\mathbf{s}') \cup \left(\bigcup_{(a,b): \mathcal{K}_{a,b}^{L,(i,j)} \neq \emptyset} E_{a,b}(\mathbf{s}') \right) \right\}, \quad (22)$$

with respect to the distribution $W^n(\mathbf{z}|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s})$. $E_0(\mathbf{s}')$ is the set of all $\mathbf{z} \in \mathcal{Z}^n$ for which (6) is violated with $\mathbf{x}_i, \mathbf{y}_j, \mathbf{x}_{I_K \setminus \{i\}}, \mathbf{y}_{J_K \setminus \{j\}}, \mathbf{s}', \mathbf{z}$ and each of the $E_{a,b}(\mathbf{s}')$ is the set of all $\mathbf{z} \in \mathcal{Z}^n$ for which (7) is violated with $\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}', \mathbf{z}$ for some $K \in \mathcal{K}_{a,b}^{L,(i,j)}$, where $\mathcal{K}_{a,b}^{L,(i,j)} \triangleq \{K \subseteq [M] \times [M], (i,j) \in K, |K| = L+1, |I_K| = a, |J_K| = b\}$. As

$$E_0(\mathbf{s}) \cup \left(\bigcup_{(a,b): \mathcal{K}_{a,b}^{L,(i,j)} \neq \emptyset} E_{a,b}(\mathbf{s}) \right) \quad (23)$$

subsumes (22), it suffices to prove the exponential decays of $W^n(E_0(\mathbf{s}))|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}$ and $W^n(E_{a,b}(\mathbf{s}))|\mathbf{x}_i, \mathbf{y}_j, \mathbf{s}$, for every \mathbf{s} , $(i, j) \in \mathcal{A}_{\epsilon}(\mathbf{s}) \cap [\mathcal{B}_{\epsilon}(\mathbf{s}) \times \mathcal{C}_{\epsilon}(\mathbf{s})]$ and (a, b) such that $\mathcal{K}_{a,b}^{L,(i,j)} \neq \emptyset$. These proofs follow using the well-known techniques from the method of types [4]. Specifically, the former follows exactly as in (35) of [2] using (16), the complement of (6) and (21), while the latter is a straightforward modification of (40) and (46) of [2] using (17), (18), the complement of (7) and (21). The details are omitted. ■

V. DISCUSSION

At present, there is a gap between Theorem 2 and Theorem 3, i.e., there exists a range of list sizes for which we cannot determine C_L . This is caused by the fact that our present definition of symmetrizability only captures the “shapes” of an A -diagonal (3) and an A -rectangle (4), while the output of a list decoder (for a fixed received sequence) can have any “shape.” It remains to capture the other shapes in a “single-letter” form.

The generalization to the case of an AVMAC with m -transmitters is fairly straightforward. In particular, Definition 1 can be modified to take care of A -diagonals and A -rectangles in the m -product space of indices. In addition, $g_m(A)$ (as in (8)) can also be defined as the smallest cardinality of a subset of the m -product space of indices which always contains either an A -diagonal or an A -rectangle. In general, $g_m(A) \leq (A-1)^m + 1$. By suitably generalizing the list decoding scheme and Lemma 4 in the proof of Theorem 3, the latter can be generalized for an AVMAC with m -transmitters which will give a lower bound $= (U+1)^m$ for the list size above which $C_L = C^R$.

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